

L^2 -ESTIMATES FOR THE d -OPERATOR ACTING ON SUPER FORMS

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ABSTRACT. In the setting of super forms developed in [7], we introduce the notion of \mathbb{R} -Kähler metrics on \mathbb{R}^n . We consider existence theorems and L^2 -estimates for the equation $d\alpha = \beta$, where α and β are super forms, in the spirit of Hörmander's L^2 -estimates for the $\bar{\partial}$ -equation on a complex Kähler manifold.

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1. INTRODUCTION

This article is concerned with introducing the notion of an \mathbb{R} -Kähler metric on the Euclidean space, \mathbb{R}^n . Let us explain the meaning of this statement: on a complex manifold, a hermitian metric induces a $(1,1)$ -form ω , and the manifold is *Kähler* if $d\omega = 0$. In [7] the formalism of super forms on \mathbb{R}^n was considered, which enables us to define (p,q) -forms on \mathbb{R}^n . In particular, a smooth metric g on \mathbb{R}^n can be represented by a smooth, positive $(1,1)$ -form ω , and in analogy with the complex setting, we define the metric g to be *\mathbb{R} -Kähler* if $d\omega = 0$. In this article, our main concern is for the d -equation for (p,q) -forms on \mathbb{R}^n endowed with a Kähler metric; by this we mean that given a (p,q) -form β , we wish to find a $(p-1,q)$ -form α solving the equation

$$d\alpha = \beta.$$

Under certain hypothesis on β , we shall prove existence theorems for this equation using arguments from the technique of L^2 -estimates due to Hörmander for the $\bar{\partial}$ -equation on a complex Kähler manifold (c.f. [6]). This will also give us an L^2 -estimate on the solution α in terms of β on a given L^2 -space (depending on the Kähler metric), to be introduced later in this article. As a particular case, we are able to solve the d -equation for ordinary p -forms on \mathbb{R}^n together with an L^2 -estimate on the solution in terms of the given data. The key point in

applying the arguments of Hörmander is to establish a Kodaira-Bochner-Nakano-type identity (c.f [8]) for natural Laplace-operators arising in our setting. We also take the opportunity to introduce, in analogy with the complex case, the theory of primitive super forms. Our hope is that the results developed in this article can be used to establish results in convex analysis. For instance, there are many articles concerned with convex inequalities that utilizes L^2 -theory (see for instance [3],[1]), and we hope that our approach in this article will give a fruitful addition to the theory already developed.

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2. PRELIMINARIES

In this article, we will consider differential forms in $\mathbb{R}^n \times \mathbb{R}^n = \{(x_1, \dots, x_n, \xi_1, \dots, \xi_n)\}$ with coefficients depending only on the variables (x_1, \dots, x_n) . Such forms, which we shall call super forms, were considered in the article [7]. We say that α is a (p, q) -form if

$$\alpha = \sum_{|I|=p, |J|=q} \alpha_{IJ}(x) dx_I \wedge d\xi_J,$$

where we use multi-index notation, and a k -form is a (p, q) -form with $p + q = k$. The set of (p, q) -forms whose coefficients are smooth will be denoted by $\mathcal{E}^{p,q}$. A smooth $(1, 1)$ -form ω is said to be *positive* if the coefficient-matrix $(\omega_{ij}(x))_{i,j}$ is positive definite, for each x . Let us define the operator

$$d^\# : \mathcal{E}^{p,q} \rightarrow \mathcal{E}^{p,q+1}$$

by letting

$$d^\# \left(\sum_{|I|=p, |J|=q} \alpha_{IJ}(x) dx_I \wedge d\xi_J \right) = \sum_{i=1}^n \sum_{|I|=p, |J|=q} \frac{\partial}{\partial x_i} \alpha_{IJ}(x) d\xi_i \wedge dx_I \wedge d\xi_J.$$

The operator $d : \mathcal{E}^{p,q} \rightarrow \mathcal{E}^{p+1,q}$ is defined as usual, and a form α is called *closed* if $d\alpha = 0$. We also define the linear map

$$J : \{(p, q) - \text{forms}\} \rightarrow \{(q, p) - \text{forms}\}$$

by letting

$$J \left(\sum_{|I|=p, |J|=q} \alpha_{IJ}(x) dx_I \wedge d\xi_J \right) = \sum_{|I|=p, |J|=q} \alpha_{IJ}(x) d\xi_I \wedge dx_J.$$

Observe that this makes for $J^2 = Id$. The operator $d^\#$ can be written in terms of J as

$$d^\# = J \circ d \circ J.$$

We have the following result (cf. [7]):

Proposition 2.1. *A closed, smooth $(1, 1)$ -form ω is positive if and only if there exists a convex, smooth function f such that*

$$\omega = dd^\# f.$$

Now fix a smooth, positive, and closed $(1, 1)$ -form ω . We shall use the notation

$$\omega_q = \omega^q / q!$$

Such a form ω induces a metric on \mathbb{R}^n in a natural way: if $v = (v_1, \dots, v_n), w = (w_1, \dots, w_n) \in \mathbb{R}^n$, then for every $x \in \mathbb{R}^n$, we define

$$(v, w)_x = \sum_{i,j=1}^n v_i w_j \omega_{ij}(x),$$

where the functions ω_{ij} are defined by $\omega = \sum_{i,j=1}^n \omega_{ij} dx_i \wedge d\xi_j$. We obtain an induced metric on the space of $(1, 0)$ - and $(0, 1)$ -forms: if $\alpha = \sum \alpha_i dx_i$ then $(\alpha, \alpha)_x = \sum \omega^{ij}(x) \alpha_i \alpha_j$ where (ω^{ij}) denotes the inverse of the matrix (ω_{ij}) , and analogously for $(0, 1)$ -forms. Using this metric, we would like to define the norm of a (p, q) -form, at a point. Let us fix an orthonormal (with respect to ω) coordinate system (dx_1, \dots, dx_n) for the space of $(1, 0)$ -forms. If $\alpha = \sum \alpha_{IJ} dx_I \wedge d\xi_J$, we define

$$(2.1) \quad |\alpha|^2 = \sum |\alpha_{IJ}|^2.$$

If $\alpha = \sum_{|I|=p} \alpha_I dx_I$ and $\beta = \sum_{|J|=q} \beta_J d\xi_J$, then

$$|\alpha \wedge \beta|^2 = \sum_{I,J} \alpha_I^2 \beta_J^2 = |\alpha|^2 |\beta|^2.$$

If we polarize this formula we obtain

$$(2.2) \quad (\alpha \wedge \beta, \alpha' \wedge \beta') = (\alpha, \alpha')(\beta, \beta'),$$

with $(p, 0)$ -forms α, α' and $(0, q)$ -forms β, β' , and where (\cdot, \cdot) denotes the inner product associated with the norm $|\cdot|$. Let us show that the definition (2.1) is independent of the choice of orthonormal coordinate system: We begin with the case of $(p, 0)$ -forms: Let $\alpha = \sum_{|I|=p} \alpha_I dx_I$. A simple calculation shows that,

$$|\alpha|^2 \omega_n = c_p \alpha \wedge J(\alpha) \wedge \omega_{n-p},$$

where $c_p = (-1)^{p(p-1)/2}$, and this expression does not depend on the basis chosen. The number c_p is chosen such that $dx_{i_1} \wedge \dots \wedge dx_{i_p} \wedge d\xi_{i_1} \wedge \dots \wedge d\xi_{i_p} = c_p \cdot dx_{i_1} \wedge d\xi_{i_1} \wedge \dots \wedge dx_{i_p} \wedge d\xi_{i_p}$. The same calculations hold for $(0, q)$ -forms. Thus, at least for $(p, 0)$ - and $(0, q)$ -forms, formula (2.1) does not depend on which orthonormal coordinates we choose. Now, let (dy_1, \dots, dy_n) be another orthonormal basis, and let $d\xi_i = J(dy_i)$. If $\alpha = \sum \alpha_{IJ} dy_I \wedge d\xi_J$, then by (2.2) we get that

$$(\alpha, \alpha) = \sum \alpha_{IJ} \alpha_{KL} (dy_I, dy_K)(d\xi_J, d\xi_L).$$

But by the above, we know that (\cdot, \cdot) does not depend on which orthonormal basis we work with, when applied to $(p, 0)$ - or $(0, q)$ -forms. Thus (dy_I, dy_K) and $(d\xi_J, d\xi_L)$ is non zero, and equal to one, if and only if $I = K$ and $J = L$. Thus the definition is independent of which orthonormal basis we use. When we wish to emphasize which metric ω the norm and inner product depend on, we will write $|\cdot|_\omega$, and $(\cdot, \cdot)_\omega$.

The Hodge-star in our setting is defined by the relation

$$(2.3) \quad \alpha \wedge *J(\beta) = (\alpha, \beta) \omega_n.$$

For an example, if we choose orthonormal coordinates at a point, then in terms of these we have that

$$*dx_I \wedge d\xi_J = c_{IJ} \cdot dx_{J^c} \wedge d\xi_{I^c},$$

for a constant $c_{IJ} = \pm 1$ chosen so that (2.3) is true; here I^c denotes the complementary index of I . We will later investigate the constant c_{IJ} more carefully.

The integral of an (n, n) -form $\alpha = \alpha_0(x)c_n dx \wedge d\xi$, is defined by

$$(2.4) \quad \int_{\mathbb{R}^n \times \mathbb{R}^n} \alpha_0(x)c_n dx \wedge d\xi = \int_{\mathbb{R}^n} \alpha_0(x)dx,$$

and this gives us an L^2 -structure on the space of forms:

$$L_{p,q}^2 = \{(p, q) - \text{forms } \alpha : \int_{\mathbb{R}^n \times \mathbb{R}^n} |\alpha|^2 \omega_n < +\infty\}.$$

We will later consider a weighted version of this L^2 -space. We remark that in defining the integral (2.4) we have fixed a volume element $d\xi$ on which the integral thus depends.

3. COMPARISON WITH THE COMPLEX THEORY

In this section, we will consider how super forms correspond to complex forms. Let us begin in the linear setting, that is, we consider only forms at a single point, say $x_0 \in \mathbb{R}^n$. Let ω be an \mathbb{R} -Kähler form. At the point x_0 , we choose coordinates $(x_1, \dots, x_n, \xi_1, \dots, \xi_n)$ for $\mathbb{R}^n \times \mathbb{R}^n$ such that

$$\omega(x_0) = \sum_{k=1}^n dx_k \wedge d\xi_k.$$

Since we will consider complex forms as well, we let (z_1, \dots, z_n) be the standard complex coordinates of \mathbb{C}^n . We will use the notation,

$$dV_i = dx_i \wedge d\xi_i, \quad dV_i^{\mathbb{C}} = dz_i \wedge d\bar{z}_i$$

and for a multi-index $I = (i_1, \dots, i_p)$, we let

$$dV_I = dV_{i_1} \wedge \dots \wedge dV_{i_p}, \quad dV_I^{\mathbb{C}} = dV_{i_1}^{\mathbb{C}} \wedge \dots \wedge dV_{i_p}^{\mathbb{C}}.$$

Now, define

$$\Theta_{I,J,K} = dx_J \wedge d\xi_K \wedge dV_I,$$

for disjoint indices I, J , and K . We also define the complex form

$$\Theta_{I,J,K}^{\mathbb{C}} = dz_J \wedge d\bar{z}_K \wedge dV_I^{\mathbb{C}}.$$

Every super form α can at a fixed point be written as a linear combination

$$\alpha = \sum \alpha_{I,J,K} \Theta_{I,J,K},$$

where the coefficients $\alpha_{I,J,K}$ are real numbers; we define a map \mathcal{C} which takes super forms to complex forms by,

$$(3.1) \quad \mathcal{C}(\alpha) = \sum \alpha_{I,J,K} \Theta_{I,J,K}^{\mathbb{C}}.$$

The map \mathcal{C} is linear by definition, and it is also injective, since $\alpha = 0$ is equivalent to $\mathcal{C}(\alpha) = 0$. However, only complex forms of the type (3.1) with real coefficients correspond to a super form, and thus, the correspondence describes an isomorphism between the vector space of super forms at a fixed point and the vector space of complex forms of the form (3.1) with real coefficients $\alpha_{I,J,K}$. This latter space is, from the complex point of view, not very natural and depends very much on the choice of coordinates. For instance, a generic change of coordinates on the complex side does not leave this space invariant.

The operation of multiplying with the \mathbb{R} -Kähler form ω is sufficiently important to deserve its own notation:

Definition 3.1. We define the operator

$$L : \{k\text{-forms}\} \rightarrow \{(k+2)\text{-forms}\}$$

by letting

$$L(\alpha) = \omega \wedge \alpha,$$

where α is a k -form. The dual Λ of the operator L is defined by,

$$(L(\alpha), \beta) = (\alpha, \Lambda(\beta)),$$

for β a $(k+2)$ -form.

On the complex side, we set the Kähler form to be $\Omega = \frac{i}{2} \sum_{k=1}^n dz_k \wedge d\bar{z}_k$, with corresponding operator L_Ω . This form Ω induces an inner product on the space of complex forms such that the square of the norm of $\sum \alpha_{I,J,K} \Theta_{I,J,K}^{\mathbb{C}}$ is equal to $\sum |\alpha_{I,J,K}|^2$ in exactly the same way as in formula (2.1). The definitions are made so that, if α is a super form, then the norm of $\mathcal{C}(\alpha)$ measured with respect to Ω , is equal to the norm of α measured with respect to ω . Thus the correspondence $\alpha \leftrightarrow \alpha^{\mathbb{C}}$ is in fact an isometry. We denote by Λ_Ω the dual of L_Ω with respect to the metric given by Ω . We have the following:

Proposition 3.2. Let α be a k -form. Then

$$(3.2) \quad \mathcal{C}(L\alpha) = \frac{2}{i} L_\Omega(\mathcal{C}(\alpha)),$$

$$(3.3) \quad \mathcal{C}(\Lambda\alpha) = \frac{i}{2} \Lambda_\Omega(\mathcal{C}(\alpha)).$$

Moreover,

$$\Lambda\alpha = 0 \iff \Lambda_\Omega \mathcal{C}(\alpha) = 0.$$

Proof. We let $I+i$ be the multi index $I \cup \{i\}$ and $I-i = I \setminus \{i\}$ which we define to be the empty set if $i \notin I$. First, the formula (3.2) is immediate. Next, we claim that

$$(3.4) \quad \Lambda \Theta_{L,M,N} = \sum_{j \in L} \Theta_{L-j,M,N},$$

where we use the convention that if an index I, J , or K is the empty set, then $\Theta_{I,J,K} = 0$. One realizes this as follows: we have that

$$L \Theta_{I,J,K} = \sum_{\{i \notin I \cup J \cup K\}} \Theta_{I+i,J,K}.$$

and that Λ is defined by the relation

$$(L(\Theta_{I,J,K}), \Theta_{L,M,N}) = (\Theta_{I,J,K}, \Lambda(\Theta_{L,M,N})).$$

Here the left hand side is non-zero if and only if there is an $i \notin I \cup J \cup K$, such that $I+i = L$ and $J = M, K = N$. In this case, the left hand side is equal to 1, which proves the formula. On the other hand, we have the well known formula (c.f [9], p.21)

$$\Lambda_\Omega \Theta_{L,M,N}^{\mathbb{C}} = \frac{2}{i} \sum_{j \in L} \Theta_{L-j,M,N}^{\mathbb{C}}.$$

Thus, using linearity of Λ , we conclude that formula (3.3) holds. The last part follows since $\Lambda\alpha = 0 \iff \mathcal{C}(\Lambda\alpha) = 0 \iff \Lambda_\Omega \mathcal{C}(\alpha) = 0$. \square

The Hodge-star $*_\Omega$, acting on complex forms, is defined by the formula

$$v \wedge *_\Omega(\bar{v}) = |v|^2 \Omega_n.$$

Let

$$N = \{1, 2, \dots, n\},$$

and recall that we defined

$$c_p = (-1)^{p(p-1)/2},$$

for each integer p ; the number c_p was chosen so that

$$dx_I \wedge d\xi_I = c_p dx_{i_1} \wedge d\xi_{i_1} \wedge \dots \wedge dx_{i_p} \wedge d\xi_{i_p}$$

where $I = (i_1, \dots, i_p)$. Now, it is well known that (c.f. [9], p.20) that

$$*_\Omega dz_A \wedge d\bar{z}_B \wedge dV_M^{\mathbb{C}} = \left[i^{p-q} (-1)^{k(k-1)/2+m} (-2i)^{k-n} \right] dz_A \wedge d\bar{z}_B \wedge dV_{M'}^{\mathbb{C}},$$

with $M' = N \setminus (A \cup B \cup M)$. However, a small calculations reveals that

$$*dx \wedge d\xi_B \wedge dV_M = c_p c_q (-1)^{p+m+pq} dx_A \wedge d\xi_B \wedge dV_{M'}.$$

Thus, the real and the complex Hodge stars are related by

$$(3.5) \quad \mathcal{C}(*dx \wedge d\xi_B \wedge dV_M) = i^n 2^{n-k} (-1)^n \cdot (*_\Omega dz_A \wedge d\bar{z}_B \wedge dV_M^{\mathbb{C}})$$

since a straightforward calculation shows that

$$\frac{c_p c_q (-1)^{p+m+pq}}{i^{p-q} (-1)^{k(k-1)/2+m} (-2i)^{k-n}} = i^n 2^{n-k} (-1)^n.$$

Thus, if α is a k -form, then

$$\mathcal{C}(*\alpha) = (i^n 2^{n-k} (-1)^n) *_\Omega (\mathcal{C}(\alpha)).$$

From the complex theory, if v is a complex form, there is a relation between $*^\Omega L_\Omega^r v$ and $L_\Omega^{n-r-k} v$ given by the following theorem (cf. [9], Theorem 2):

Theorem 3.3. *If $v = \sum_{|I|=p, |J|=q, |M|=m} v_{I,J,M} \Theta_{I,J,M}^{\mathbb{C}}$, then*

$$*_\Omega L_\Omega^r v = i^{p-q} (-1)^{k(k+1)/2} \frac{r!}{(n-k-r)!} L_\Omega^{n-r-k} v.$$

If we apply the above theorem to $\mathcal{C}(\alpha)$ for a k -form

$$\alpha = \sum_{|I|=p, |J|=q, |M|=m} \alpha_{I,J,M} \Theta_{I,J,M},$$

using (3.5) and that $L_\Omega^N \mathcal{C}(\alpha) = (\frac{i}{2})^N \mathcal{C}(L\alpha)$, we obtain,

$$\frac{1}{i^n 2^{n-k-2r} (-1)^n} * \left(\left(\frac{i}{2} \right)^r L^r \alpha \right) = i^{p-q} (-1)^{k(k+1)/2} \frac{r!}{(n-k-r)!} \left(\frac{i}{2} \right)^{n-k-r} L^{n-r-k} \alpha,$$

which gives us

$$*L^r \alpha = (-1)^{k(k+1)/2+r+q+m} \frac{r!}{(n-k-r)!} L^{n-r-k} \alpha.$$

Thus, we have proved the following:

Theorem 3.4. *If*

$$\alpha = \sum_{|I|=p, |J|=q, |M|=m} \alpha_{I,J,M} \Theta_{I,J,M},$$

and $k = p + q + 2m$, then

$$(3.6) \quad *L^r \alpha = (-1)^{k(k+1)/2+r+q+m} \frac{r!}{(n-k-r)!} L^{n-r-k} \alpha.$$

This far, we have only compared super forms with complex forms in the linear setting, that is, at a fixed point. Let us now extend the map \mathcal{C} to be defined on super forms on all of \mathbb{R}^n . Until this point, there has been no need for a relationship between our real coordinates $(x_1, \dots, x_n, \xi_1, \dots, \xi_n)$ and (z_1, \dots, z_n) , but now we make the usual identification $z_k = x_k + iy_k$ for each $k = 1, \dots, n$. For

$$\alpha(x) = \sum \alpha_{IJM}(x) \Theta_{IJM},$$

where $\alpha_{IJM}(\cdot)$ are functions on \mathbb{R}^n , we define

$$\mathcal{C}(\alpha)(z) = \sum \alpha_{IJM}(x) \Theta_{IJM}^{\mathbb{C}},$$

where $x = (z + \bar{z})/2$.

Proposition 3.5. *For any super form α we have that*

$$\mathcal{C}(d\alpha) = 2\partial\mathcal{C}(\alpha),$$

$$\mathcal{C}(d^{\#}\alpha) = 2\bar{\partial}\mathcal{C}(\alpha).$$

Proof. Let $\alpha(x) = \sum_{I,J,M} \alpha_{IJM}(x) dx_I \wedge d\xi_J \wedge dV_M$. Then

$$\begin{aligned} \mathcal{C}(d\alpha) &= \mathcal{C}\left(\sum_{I,J,M,l} \frac{\partial \alpha_{IJM}(x)}{\partial x_l} dx_l \wedge dx_I \wedge d\xi_J \wedge dV_M\right) = \\ &= \sum_{I,J,M,l} \frac{\partial \alpha_{IJM}(x)}{\partial x_l} dz_l \wedge dz_I \wedge d\bar{z}_J \wedge dV_M^{\mathbb{C}}. \end{aligned}$$

Since $\frac{\partial}{\partial z_l} = \frac{1}{2}\left(\frac{\partial}{\partial x_l} - i\frac{\partial}{\partial y_l}\right)$, we see that

$$\frac{\partial \alpha(x)}{\partial x_l} = 2 \frac{\partial \alpha(x)}{\partial z_l},$$

and thus

$$\mathcal{C}(d\alpha) = 2\partial(\mathcal{C}(\alpha)).$$

The formula for $\bar{\partial}$ follows in the same way. □

An important formula in complex analysis is the following (c.f [9], p. 42-44):

Theorem 3.6. *For any complex form v ,*

$$[\Lambda_{\Omega}, \partial]v = -i *_{\Omega} \partial *_{\Omega} v.$$

Let α be a k -form. Then, applying the above theorem, we get

$$[\Lambda_\Omega, \partial]\mathcal{C}(\alpha) = -i *_\Omega \partial *_\Omega \mathcal{C}(\alpha).$$

However, by Propositions 3.2 and 3.5, we notice that

$$[\Lambda_\Omega, \partial]\mathcal{C}(\alpha) = -i \cdot \mathcal{C}([\Lambda, d]\alpha),$$

and, by repeated use of (3.5), keeping in mind that $d * \alpha$ is a $(2n - k + 1)$ -form,

$$\begin{aligned} *_\Omega \partial *_\Omega \mathcal{C}(\alpha) &= ((i^n 2^{n-k} (-1)^n)^{-1}) *_\Omega \partial \mathcal{C}(*\alpha) = \\ &= \left((i^n 2^{n-k} (-1)^n i^n 2^{n-(2n-k+1)} (-1)^n)^{-1} \right) \mathcal{C}\left(*\frac{1}{2}d(*\alpha)\right) = (-1)^n \mathcal{C}(*d * \alpha). \end{aligned}$$

This gives us that

$$-i(-1)^n \mathcal{C}(*d * \alpha) = -i \cdot \mathcal{C}([\Lambda, d]\alpha),$$

and so we arrive at:

Theorem 3.7. *For any form α we have*

$$[\Lambda, d]\alpha = (-1)^n * d * (\alpha).$$

Let us conclude this section with some elementary observations:

Lemma 3.8. *For any k -form α we have*

$$(3.7) \quad \begin{aligned} * * \alpha &= (-1)^{n-k} \alpha \\ \Lambda J \alpha &= -J \Lambda \alpha, \end{aligned}$$

and

$$*J = (-1)^n J *.$$

Proof. Since every form α is a linear combination of forms of the type $dx_A \wedge d\xi_B \wedge dV_M$, we need only to prove the lemma with $\alpha = dx_A \wedge d\xi_B \wedge dV_M$. One easily calculates

$$*dx_A \wedge d\xi_B \wedge dV_M = c_p c_q (-1)^{p+m+qp} dx_A \wedge d\xi_B \wedge dV_{M'},$$

with $M' = \{1, 2, \dots, n\} \setminus A \cup B \cup M$, and $p = |A|$, $q = |B|$, $m = |M|$ using the same notation as before. Thus by applying the Hodge-star twice, the form $dx_A \wedge d\xi_B \wedge dV_M$ will be multiplied by the constant

$$c_p^2 c_q^2 (-1)^{p+m+qp+p+m'+pq} = (-1)^{m+n-m-p-q} = (-1)^{n-k},$$

where $m' = |M'| = n - p - q - m$, which proves the first formula. The second formula follows by using (3.4). The last formula follows from direct calculations:

$$J * dx_A \wedge d\xi_B \wedge dV_M = c_p c_q (-1)^{p+m+m'} dx_A \wedge d\xi_B \wedge dV_{M'},$$

$$*J(dx_A \wedge d\xi_B \wedge dV_M) = c_p c_q (-1)^q dx_A \wedge d\xi_B \wedge dV_{M'}.$$

Thus $*J$ differs from $J*$ by the constant

$$c_p c_q (-1)^{p+m+m'} \cdot c_p c_q (-1)^q = (-1)^n.$$

□

Finally, we note the following corollary of Theorem 3.7:

Corollary 3.9. *For a form α , we have,*

$$*d^\# * \alpha = (-1)^{n+1} [\Lambda, d^\#] \alpha.$$

Proof. Let us consider the expression $J[\Lambda, d]J\alpha$. From the definition of $d^\#$ we get:

$$\begin{aligned} J[\Lambda, d]J\alpha &= J\Lambda dJ\alpha - Jd\Lambda J\alpha = J\Lambda Jd^\#\alpha + JdJ\Lambda\alpha = \\ &= -J^2\Lambda d^\#\alpha + J^2d^\#\Lambda\alpha = -[\Lambda, d^\#]\alpha, \end{aligned}$$

where we used that

$$J\Lambda = -\Lambda J.$$

On the other hand, Theorem 3.7 says that

$$J[\Lambda, d]J\alpha = (-1)^n J(*d*)J\alpha,$$

and since $*d*J = J*d^\#*$ by applying Lemma 3.8, we have proved that indeed,

$$*d^\#*\alpha = (-1)^{n+1}[\Lambda, d^\#]\alpha.$$

□

4. PRIMITIVE SUPER FORMS

In this section we take the opportunity to introduce the notion of primitivity for super forms and establish expected results by once again comparing with the complex setting.

Proposition 4.1. *Let α be a (p, q) -form with $p + q = k$. Then*

$$(4.1) \quad [\Lambda, L^s]\alpha = C_{k,s}L^{s-1}\alpha,$$

with

$$C_{k,s} = s(n - k + 1 - s).$$

Proof. The result follows from the complex theory (c.f [9]): if v is any complex form then

$$[\Lambda_\Omega, L_\Omega^s]v = C_{k,s}L_\Omega^{s-1}v,$$

and the result follows by letting $v = \mathcal{C}(\alpha)$ and by repeatedly applying Proposition 3.2. □

Let us define an important concept in this setting:

Definition 4.2. A form α is called *primitive* if α satisfies

$$\Lambda\alpha = 0.$$

Note that, in view of Proposition 3.2, α is primitive if and only if $\mathcal{C}(\alpha)$ is primitive (a complex form v is primitive if $\Lambda_\Omega v = 0$). The importance of primitive forms is that they are easier to work with than just any arbitrary form, combined with the fact that any form can be decomposed into primitive components in the following sense:

Proposition 4.3. *Let α be a k -form. Then we can write α as*

$$\alpha = \alpha_0 + L\alpha_1 + \dots + L^s\alpha_s,$$

where each α_j is a primitive $(k - 2j)$ -form. Moreover, the terms of the sum are pairwise orthogonal.

Proof. The result is well known in the complex case (c.f [9]). Thus we know that the formula holds for $\mathcal{C}(\alpha)$, that is

$$\mathcal{C}(\alpha) = \alpha'_0 + L\alpha'_1 + \dots + L^s\alpha'_s$$

where each α'_j is a primitive, complex, $(k - 2j)$ -form. But since $\mathcal{C}(\alpha)$ has real coefficients we can assume that each α'_j have real coefficients as well. Thus, each α'_j is in fact $\mathcal{C}(\alpha_j)$ for some super form α_j . By Proposition 3.2, each α_j is primitive, and the property of being pairwise orthogonal is immediate since the correspondence $\alpha \longleftrightarrow \mathcal{C}(\alpha)$ is an isometry. \square

Proposition 4.4. *Let α be a k -form. If*

$$L^{n-k+s}\alpha = 0$$

then

$$\alpha = \sum_{0 \leq j \leq s-1} L^j \alpha_j$$

with α_j a primitive $(k - 2j)$ -forms. Moreover, α is primitive iff

$$L^{n-k+1}\alpha = 0.$$

Proof. The formulas are well known in the complex case and translates into our setting in the same way as above. \square

The main theorem of this section, known in the complex case as the Lefschetz isomorphism Theorem, is given by the following:

Theorem 4.5. *Let $k \leq n$. Then the operator*

$$L^{n-k} : \{k\text{-forms}\} \rightarrow \{(2n-k)\text{-forms}\}$$

is an isomorphism.

Proof. From the complex setting, we know that $(L_\Omega)^{n-k}$ is an isomorphism, and it is easily verified that k -forms that are real linear combinations of $\Theta_{I,J,K}^C$ correspond, via L_Ω^{n-k} , to $(2n - k)$ -forms that are real linear combinations of the same type of degree $(2n - k)$ which establishes the Theorem. \square

5. L^2 -ESTIMATES FOR THE d -OPERATOR

Let us fix a closed, strictly positive, smooth $(1, 1)$ -form ω . As we have seen, ω induces an inner product on the space of forms, so that for (p, q) -forms α and β , the function $x \mapsto (\alpha, \beta)(x)$, is a function on \mathbb{R}^n . We now define the associated L^2 inner product:

Definition 5.1. For (p, q) -forms α and β we define

$$\langle \alpha, \beta \rangle = \int_{\mathbb{R}^n \times \mathbb{R}^n} (\alpha, \beta) \omega_n$$

and we define the associated norm

$$\|\alpha\|^2 = \langle \alpha, \alpha \rangle.$$

Observe that, by (2.3), we have that

$$(5.1) \quad \langle \alpha, \beta \rangle = \int_{\mathbb{R}^n \times \mathbb{R}^n} (\alpha, \beta) \omega_n = \int_{\mathbb{R}^n \times \mathbb{R}^n} \alpha \wedge *J(\beta).$$

We defined before the space $L^2_{p,q}$ as the set of all (p, q) -forms α , whose coefficients are integrable, and which satisfies $\|\alpha\|^2 < \infty$. Moreover, we let

$$L^2 = \oplus_{p,q=0}^n (L^2_{p,q}).$$

We consider the operator $d : L^2_{p,q} \rightarrow L^2_{p+1,q}$ as a closed, densely defined operator, with

$$\text{dom}(d) = \{\alpha \in L^2_{p,q} : d\alpha \in L^2_{p+1,q}\},$$

where d is taken in sense of super currents if α is not smooth (cf. [7]). If α is an (n, q) -form it is understood that $d\alpha = 0$. By standard arguments, smooth (p, q) -forms with compact support is dense in $L^2_{p,q}$ and each such form is in $\text{dom}(d)$. Thus, $\text{dom}(d)$ is indeed dense in $L^2_{p,q}$. We define the dual of the operator d with respect to the inner product by, the relation

$$\langle d^* \alpha, \beta \rangle = \langle \alpha, d\beta \rangle$$

for smooth forms $\alpha, \beta \in L^2$. The dual of $d^\#$ is defined analogously.

Proposition 5.2. *For a smooth, compactly supported form α we have*

$$d^* \alpha = [\Lambda, d^\#] \alpha,$$

and

$$(d^\#)^* \alpha = -[\Lambda, d] \alpha.$$

Proof. By Theorem 3.7 and Corollary 3.9 it is enough to prove that

$$d^* = (-1)^{n+1} * d^\# *$$

and

$$(d^\#)^* = (-1)^n * d *.$$

To this end, let α be a k -form and β a $(k+1)$ -form, both smooth and compactly supported. By (5.1) and Stokes' formula,

$$\langle d\alpha, \beta \rangle = \int d\alpha \wedge *J(\beta) = (-1)^{k+1} \int \alpha \wedge d * J(\beta).$$

Since $d * J(\beta)$ is a $(2n-k)$ -form, we know from Lemma 3.8 that

$$d * J(\beta) = (-1)^{n-(2n-k)} * d * J(\beta) = (-1)^{n-(2n-k)} * J(*d^\# * \beta),$$

since $dJ = Jd^\#$. Thus,

$$\langle d\alpha, \beta \rangle = (-1)^{k+1+n-(2n-k)} \int \alpha \wedge * d * J(\beta) = (-1)^{n+1} \langle \alpha, *d^\# * \beta \rangle,$$

which proves that

$$d^* \beta = (-1)^{n+1} * d^\# * \beta.$$

By Corollary 3.9, we see that indeed

$$d^* = [\Lambda, d^\#].$$

The second formula of the Proposition follows in the same way, using Theorem 3.7. \square

There are two natural Laplace operators in our setting:

Definition 5.3. For α any smooth and compactly supported form, we define

$$\square\alpha = dd^*\alpha + d^*d\alpha$$

and

$$\square^\# \alpha = d^\#(d^\#)^*\alpha + (d^\#)^*d^\#\alpha.$$

Our previous work can now be applied to show that these operators are in fact equal:

Proposition 5.4. *For any smooth and compactly supported form α we have that*

$$\square\alpha = \square^\#\alpha.$$

Proof. By Proposition 5.2, we obtain

$$\square\alpha = (d[\Lambda, d^\#]\alpha + [\Lambda, d^\#]d\alpha)$$

and

$$\square^\#\alpha = -(d^\#[\Lambda, d]\alpha + [\Lambda, d]d^\#\alpha).$$

Writing out the terms explicitly one immediately concludes that these expressions are equal. \square

Let us consider “twisted” versions of these Laplacians:

Definition 5.5. For φ a smooth function, we define

$$d_\varphi = e^\varphi de^{-\varphi},$$

and

$$d_\varphi^\# = e^\varphi d^\# e^{-\varphi}.$$

We define the weighted inner product

$$\langle \alpha, \beta \rangle_\varphi = \int_{\mathbb{R}^n \times \mathbb{R}^n} (\alpha, \beta) e^{-\varphi} \omega_n,$$

and let $L_\varphi^2 = \oplus_{0 \leq p, q \leq n} (L_{p, q, \varphi}^2)$ be the space of forms such that

$$\|\alpha\|_\varphi^2 := \langle \alpha, \alpha \rangle_\varphi < \infty.$$

This is easily seen to be a Hilbert space. We will write $L_\varphi^2(\omega)$ when we wish to emphasize which \mathbb{R} -Kähler metric ω we are integrating against in defining L_φ^2 . The dual of d and $d^\#$ with respect to this inner product will be denoted d^* and $(d^\#)^*$ respectively. We can now introduce the “twisted” Laplacians:

$$\square_\varphi \alpha = dd^*\alpha + d^*d\alpha$$

and

$$\square_\varphi^\# \alpha = d_\varphi^\#(d_\varphi^\#)^*\alpha + (d_\varphi^\#)^*d_\varphi^\#\alpha.$$

Our next task is to relate these Laplacians to each other in the spirit of Proposition 5.4. We begin with the weighted analogue of Proposition 5.2:

Proposition 5.6. *For any smooth, compactly supported form α , the equations*

$$d^*\alpha = [\Lambda, d_\varphi^\#]\alpha,$$

and

$$(d_\varphi^\#)^*\alpha = -[\Lambda, d]\alpha,$$

are satisfied.

Proof. Let α be a (p, q) -form and β a $(p+1, q)$ -form. Then, we compute

$$\langle d\alpha, \beta \rangle_\varphi = \int (d\alpha, \beta) e^{-\varphi} = \int (\alpha, d^*(e^{-\varphi} \beta)) = \int (\alpha, e^\varphi d^*(e^{-\varphi} \beta)) e^{-\varphi}.$$

By Proposition 5.2 we know that $d^* = [\Lambda, d^\#]$. Inserting this into the last integral, we see that

$$\langle d\alpha, \beta \rangle_\varphi = \int (\alpha, e^\varphi [\Lambda, d^\#](e^{-\varphi} \beta)) e^{-\varphi} = \int (\alpha, [\Lambda, d^\#_\varphi] \beta) e^{-\varphi},$$

using that Λ commutes with the operation of multiplying with e^φ . But this means precisely that

$$d^* \beta = [\Lambda, d^\#_\varphi] \beta,$$

which proves the first formula. The second one follows in the same way. \square

Theorem 5.7. *If α is a smooth and compactly supported form, then*

$$(5.2) \quad \square_\varphi \alpha = \square^\#_\varphi \alpha + [dd^\#_\varphi, \Lambda] \alpha.$$

Proof. We calculate, using Proposition 5.6,

$$\begin{aligned} \square_\varphi \alpha - \square^\#_\varphi \alpha &= dd^* \alpha + d^* d \alpha - (d^\#_\varphi (d^\#_\varphi)^* \alpha + (d^\#_\varphi)^* d^\#_\varphi \alpha) = \\ &= d[\Lambda, d^\#_\varphi] \alpha + [\Lambda, d^\#_\varphi] d \alpha + d^\#_\varphi [\Lambda, d] \alpha + [\Lambda, d] d^\#_\varphi \alpha. \end{aligned}$$

Since

$$d^\#_\varphi \alpha = d^\# \alpha - d^\# \varphi \wedge \alpha,$$

we get

$$\square_\varphi \alpha - \square^\#_\varphi \alpha = \square \alpha - \square^\# \alpha - d[\Lambda, d^\#_\varphi] \alpha - [\Lambda, d^\#_\varphi] d \alpha - d^\# \varphi \wedge [\Lambda, d] \alpha - [\Lambda, d] (d^\# \varphi \wedge \alpha).$$

Here we identify $d^\# \varphi$ with the operator sending $\alpha \mapsto d^\# \varphi \wedge \alpha$. By Proposition 5.4, we know that the un-weighted Laplace operators satisfy $\square \alpha - \square^\# \alpha = 0$. Thus,

$$\square_\varphi \alpha - \square^\#_\varphi \alpha = -(d[\Lambda, d^\#_\varphi] \alpha + [\Lambda, d^\#_\varphi] d \alpha + d^\# \varphi \wedge [\Lambda, d] \alpha + [\Lambda, d] (d^\# \varphi \wedge \alpha)).$$

Expanding the commutators, we see that

$$\begin{aligned} \square_\varphi \alpha - \square^\#_\varphi \alpha &= -(d(\Lambda(d^\# \varphi \wedge \alpha)) - d(d^\# \varphi \wedge (\Lambda \alpha)) + \Lambda(d^\# \varphi \wedge d \alpha) - d^\# \varphi \wedge (\Lambda d \alpha) + \\ &\quad + d^\# \varphi \wedge (\Lambda d \alpha) - d^\# \varphi (d(\Lambda \alpha)) + \Lambda(d(d^\# \varphi \wedge \alpha) - d(\Lambda(d^\# \varphi \wedge \alpha))). \end{aligned}$$

Removing the terms which cancel out, we obtain

$$\begin{aligned} \square_\varphi \alpha - \square^\#_\varphi \alpha &= d(d^\# \varphi \wedge (\Lambda \alpha)) - \Lambda(d^\# \varphi \wedge d \alpha) - \Lambda(d(d^\# \varphi \wedge \alpha) + d^\# \varphi \wedge (d(\Lambda \alpha))) = \\ &= dd^\# \varphi \wedge (\Lambda \alpha) - d^\# \varphi \wedge d(\Lambda \alpha) - \Lambda(d^\# \varphi \wedge d \alpha) - \Lambda(dd^\# \varphi \wedge \alpha) + \Lambda(d^\# \varphi \wedge d \alpha) + d^\# \varphi \wedge (d(\Lambda \alpha)) = \\ &= dd^\# \varphi \wedge (\Lambda \alpha) - \Lambda(dd^\# \varphi \wedge \alpha) = [dd^\#_\varphi, \Lambda] \alpha. \end{aligned}$$

Putting everything together, we conclude that

$$\square_\varphi \alpha = \square^\#_\varphi \alpha + [dd^\#_\varphi, \Lambda] \alpha,$$

as desired. \square

Example 5.8. Let us consider a concrete example of this identity. Let $n = 1$ and let f be a smooth function with compact support. For the weight function, we choose $\varphi = x^2/2$. Then

$$\square_\varphi f = d^* df = d^*(f' dx) = -f'' + x f'$$

and

$$\square_\varphi^\# f = (d_\varphi^\#)^* d_\varphi^\# f = (d_\varphi^\#)^* [(f' - x f) d\xi] = -f'' + f + x f'.$$

Thus

$$\square_\varphi f = \square_\varphi^\# f - f,$$

and we see that in this case $[dd^\# \varphi, \Lambda] = -Id$ as predicted by formula (5.2).

Let us take the inner product of identity (5.2) against a form smooth, compactly supported form α :

$$\langle \square_\varphi \alpha, \alpha \rangle_\varphi = \langle \square_\varphi^\# \alpha, \alpha \rangle_\varphi + \langle [\Lambda, dd^\# \varphi] \alpha, \alpha \rangle_\varphi \iff$$

$$\langle dd^* \alpha + d^* d \alpha, \alpha \rangle_\varphi = \langle d_\varphi^\# (d_\varphi^\#)^* \alpha + (d_\varphi^\#)^* d_\varphi^\# \alpha, \alpha \rangle_\varphi + \langle [dd^\# \varphi, \Lambda] \alpha, \alpha \rangle_\varphi.$$

By the definition of the adjoint, this expression gives us the following fundamental identity, which should be compared with the classical Bochner-Kodaira-Nakano identity of complex analysis (c.f [8]):

Theorem 5.9. *For every smooth, compactly supported form α ,*

$$(5.3) \quad \|d^* \alpha\|_\varphi^2 + \|d \alpha\|_\varphi^2 = \|(d_\varphi^\#)^* \alpha\|_\varphi^2 + \|d_\varphi^\# \alpha\|_\varphi^2 + \langle [dd^\# \varphi, \Lambda] \alpha, \alpha \rangle_\varphi.$$

By the following fundamental theorem of functional analysis, such an equality can be used to prove the existence of solutions of the d -equation (c.f. [6]):

Theorem 5.10. *Let E and F be two Hilbert spaces, equipped with norms $\|\cdot\|_E$ and $\|\cdot\|_F$ and let \mathcal{H} be a closed subspace of F . Let $\mathcal{L} : E \rightarrow F$ be a closed, densely defined operator such that $\text{dom}(\mathcal{L}^*)$ is dense in F , and that $\text{Range}(\mathcal{L}) \subset \mathcal{H}$. If, for each $\alpha \in \text{dom}(\mathcal{L}^*) \cap \mathcal{H}$, the inequality*

$$(5.4) \quad \|\mathcal{L}^* \alpha\|_F^2 \geq c \|\alpha\|_E^2$$

is satisfied for some fixed constant $c > 0$, then we can find an element $\beta \in E$ such that

$$\mathcal{L} \beta = \alpha$$

and

$$\|\beta\|_E^2 \leq c^{-1} \|\alpha\|_F^2.$$

To apply the above theorem to the operator d , we thus need to show an inequality of the type (5.4) with $\mathcal{L} = d$. Let us begin with:

Proposition 5.11. *Let $\alpha \in \text{dom}(d^*) \cap \text{dom}(d)$. Then, if the inequality*

$$(5.5) \quad \|d \beta\|_\varphi^2 + \|d^* \beta\|_\varphi^2 \geq c \|\beta\|_\varphi^2$$

holds for all smooth, compactly supported forms β with $c > 0$, then

$$(5.6) \quad \|d \alpha\|_\varphi^2 + \|d^* \alpha\|_\varphi^2 \geq c \|\alpha\|_\varphi^2.$$

Proof. We can of course assume that $d^*\alpha \in L^2_{p-1,q}$, since otherwise the inequality trivially holds, and the condition $\alpha \in \text{dom}(d)$ means precisely that $d\alpha \in L^2_{p+1,q}$. First, we show that if the inequality (5.5) holds for $\beta \in \text{dom}(d^*) \cap \text{dom}(d)$ with compact support, then the desired inequality (5.6) holds. Indeed, let χ_R be a smooth bump function which is 1 on the ball defined by $\{x \in \mathbb{R}^n : |x| \leq R\}$ and vanishes outside $\{x \in \mathbb{R}^n : |x| \leq 2R\}$. Then it is easy to see that $\chi_R \cdot \alpha \in \text{dom}(d^*)$. By assumption, we know that

$$\|d(\chi_R \cdot \alpha)\|_\varphi^2 + \|d^*(\chi_R \cdot \alpha)\|_\varphi^2 \geq c\|\chi_R \cdot \alpha\|_\varphi^2.$$

But $d(\chi_R \alpha) = d\chi_R \wedge \alpha + \chi_R d\alpha$, so

$$\|d(\chi_R \cdot \alpha)\|_\varphi \leq \|d\chi_R \wedge \alpha\|_\varphi + \|\chi_R d\alpha\|_\varphi.$$

The first term satisfies

$$\|d\chi_R \wedge \alpha\|_\varphi^2 = \int |d\chi_R \wedge \alpha|^2 e^{-\varphi} \omega_n \leq \int |d\chi_R|^2 |\alpha|^2 e^{-\varphi} \omega_n,$$

and by the assumptions on α , this term tend to zero by the dominated convergence theorem, since $|d\chi_R(x)| \rightarrow 0$ pointwise as $R \rightarrow 0$. Since $d\alpha$ belongs to L^2 , the second term tends to $\|d\alpha\|_\varphi^2$, and thus we see that

$$\lim_{R \rightarrow \infty} \|d(\chi_R \cdot \alpha)\|_\varphi^2 \leq \|d\alpha\|_\varphi^2.$$

For the term $d^*(\chi_R \cdot \alpha)$, a straightforward calculation reveals that

$$d^*(\chi_R \cdot \alpha) = \pm * d^\# \chi_R \wedge (*\alpha) \pm \chi_R d^* \alpha,$$

and consequently,

$$\|d^*(\chi_R \cdot \alpha)\|_\varphi \leq \|d^\# \chi_R \wedge (*\alpha)\|_\varphi + \|\chi_R d^* \alpha\|_\varphi.$$

By the same argument as above this implies that

$$\lim_{R \rightarrow \infty} \|d^*(\chi_R \cdot \alpha)\|_\varphi^2 \leq \|d^* \alpha\|_\varphi^2.$$

Combining these observations, we obtain

$$\|d\alpha\|_\varphi^2 + \|d^* \alpha\|_\varphi^2 \geq c \lim_{R \rightarrow \infty} \|\chi_R \cdot \alpha\|_\varphi^2 = c\|\alpha\|_\varphi^2,$$

as desired. The proof will thus be complete if we show that the hypothesis of the proposition implies that the inequality (5.6) holds for every $\alpha \in \text{dom}(d^*) \cap \text{dom}(d)$ with compact support. But for such an α , if we let ψ_ϵ be an approximation of the identity, it is not hard to show that

$$\alpha * \psi_\epsilon \in \text{dom}(d^*),$$

and moreover, as $\epsilon \rightarrow 0$,

$$\|d(\alpha * \psi_\epsilon)\|_\varphi^2 \rightarrow \|d\alpha\|_\varphi^2.$$

Since d^* is a first order differential operator with smooth coefficients, the same holds true for d^* in view of Friedrich's lemma (c.f. [5] or [6] Lemma 1.2.2 which applies analogously in our setting), that is,

$$\|d^*(\alpha * \psi_\epsilon)\|_\varphi^2 \rightarrow \|d^* \alpha\|_\varphi^2,$$

as $\epsilon \rightarrow 0$. Thus, since we know that (5.5) holds with $\beta = \alpha * \psi_\epsilon$, we see that the inequality (5.6) holds, and we are done. \square

Now, let φ be a smooth convex function such that $dd^\# \varphi \geq \epsilon \omega$ for some fixed $\epsilon > 0$. We claim that this implies that

$$\langle [dd^\# \varphi, \Lambda] \alpha, \alpha \rangle \geq \epsilon p \|\alpha\|^2,$$

for α a (p, n) -form. Indeed, by standard linear algebra, we can at each fixed point x_0 find orthogonal coordinates in which $\omega_{x_0} = \sum_{i=1}^n dx_i \wedge d\xi_i$ and $dd^\# \varphi_{x_0} = \sum_{i=1}^n \lambda_i dx_i \wedge d\xi_i$ where λ_i are ordered in such a way that $\lambda_1 \leq \dots \leq \lambda_n$. Since this holds for any point x_0 we can for each i consider λ_i as a function on \mathbb{R}^n which will depend continuously on the point x_0 , and by the assumption on φ we will have that $\lambda_n(x) \geq \dots \geq \lambda_1(x) \geq \epsilon$ for all x . For a (p, n) -form

$$\alpha = \sum_{|I|=p} \alpha_I dx_I \wedge d\xi,$$

a calculation reveals (c.f. [4], p. 69) that the pointwise inner product at the point x_0 satisfies

$$([dd^\# \varphi, \Lambda] \alpha, \alpha)_{x_0} = \sum_{|I|=p} \left(\sum_{i \in I} \lambda_i(x_0) \right) |\alpha_I|^2(x_0) \geq (\lambda_1(x_0) + \dots + \lambda_p(x_0)) |\alpha|^2(x_0).$$

Since x_0 was arbitrary we infer that

$$\langle [dd^\# \varphi, \Lambda] \alpha, \alpha \rangle \geq p \epsilon \|\alpha\|^2,$$

and thus, by (5.3), we obtain the inequality

$$(5.7) \quad \|d^* \alpha\|_\varphi^2 + \|d\alpha\|_\varphi^2 \geq p \epsilon \|\alpha\|_\varphi^2,$$

for every smooth, compactly supported form α . By Proposition 5.11 the above inequality will then hold for every $\alpha \in \text{dom}(d) \cap \text{dom}(d^*)$. If moreover $d\alpha = 0$, this implies that

$$\|d^* \alpha\|_\varphi^2 \geq p \epsilon \|\alpha\|_\varphi^2,$$

and so we can apply Theorem 5.10 with $\mathcal{H} = \ker(d)$ (which is a closed subspace) to obtain:

Theorem 5.12. *Let ω be an \mathbb{R} -Kähler form and let φ be a smooth function such that $dd^\# \varphi \geq \epsilon \omega$ for some $\epsilon > 0$. If $\beta \in L^2_{p,n,\varphi}$ satisfies that $d\beta = 0$ and if $p \geq 1$, then we can find an $\alpha \in L^2_{p-1,n,\varphi}$ such that*

$$d\alpha = \beta$$

and

$$\|\alpha\|_\varphi^2 \leq \frac{1}{p\epsilon} \|\beta\|_\varphi^2.$$

If we instead let the \mathbb{R} -Kähler form ω be given by $\omega = dd^\# \varphi$ for some smooth, convex function φ , then Proposition 4.1 tells us that $[dd^\# \varphi, \Lambda] \alpha = (k - n) \alpha$, if α is a k -form. Thus,

$$(5.8) \quad \|d^* \alpha\|_\varphi^2 + \|d\alpha\|_\varphi^2 \geq (k - n) \|\alpha\|_\varphi^2,$$

for every smooth, compactly supported form α and we get the following result:

Theorem 5.13. *Assume that $\omega = dd^\# \varphi > 0$ for a smooth convex function φ . If $\beta \in L^2_{p,q,\varphi}$ is a k -form with $k > n$ and $p \geq 1$ such that $d\beta = 0$, then we can find a $\alpha \in L^2_{p-1,q,\varphi}$ such that*

$$d\alpha = \beta$$

and

$$\|\alpha\|_\varphi^2 \leq \frac{1}{k-n} \|\beta\|_\varphi^2.$$

Now, if $\tilde{\beta}$ is a closed (p, n) -form, we can by the virtue of the above theorem solve the equation

$$d\tilde{\alpha} = \tilde{\beta}$$

with the estimate

$$\|\tilde{\alpha}\|_\varphi^2 \leq \frac{1}{p} \|\tilde{\beta}\|_\varphi^2.$$

Let us write this in coordinates: if $\tilde{\alpha} = \alpha \wedge d\xi$ with $\alpha = \sum_{|K|=p-1} \alpha_K dx_K$, and $\tilde{\beta} = \beta \wedge d\xi$ with $\beta = \sum_{|L|=p} \beta_L dx_L$, then

$$\int |\alpha|_{dd^\# \varphi}^2 c_n dx \wedge d\xi \leq \frac{1}{p} \int |\beta|_{dd^\# \varphi}^2 c_n dx \wedge d\xi.$$

This follows since $|\tilde{\alpha}|_{dd^\# \varphi}^2 = |\alpha|_{dd^\# \varphi}^2 \det(\omega_{ij})^{-1}$, and similarly for $|\tilde{\beta}|_{dd^\# \varphi}^2$. Let us reverse this argument: if β is a closed $(p, 0)$ -form such that $\int |\beta|_{dd^\# \varphi}^2 dx \wedge d\xi < +\infty$, we can consider the (p, n) form $\tilde{\beta} = \beta \wedge d\xi$, which will also be closed. Then

$$|\tilde{\beta}|_{dd^\# \varphi}^2 = |\beta|_{dd^\# \varphi}^2 \det(\omega_{ij})^{-1},$$

and by the above we have that $\tilde{\beta} \in L_\varphi^2$. Thus we can solve $d\tilde{\alpha} = \tilde{\beta}$, for some $(p-1, n)$ -form $\tilde{\alpha}$. But, $\tilde{\alpha} = \alpha \wedge d\xi$ for some $(p-1, 0)$ -form α , and we must have $d\alpha = \beta$. Thus we arrive at:

Theorem 5.14. *For a closed $(p, 0)$ -form β such that $\int_{\mathbb{R}^n} |\beta|_{dd^\# \varphi}^2 e^{-\varphi} dx < \infty$, we can solve*

$$d\alpha = \beta,$$

with

$$(5.9) \quad \int_{\mathbb{R}^n} |\alpha|_{dd^\# \varphi}^2 e^{-\varphi} dx \leq \frac{1}{p} \int_{\mathbb{R}^n} |\beta|_{dd^\# \varphi}^2 e^{-\varphi} dx.$$

It is interesting to note that when $p = 1$ the left-hand-side of (5.9) does not depend on the Kähler metric $dd^\# \varphi$.

Remark 5.15. Let us explain briefly how our setting is related to solving the $\bar{\partial}$ -equation on a holomorphic line bundle L over a compact Kähler manifold X (see [4] or [2] for a detailed account). Let φ be a metric on L , inducing a hermitian structure on L , and let ∇ be the Chern connection of L . Strictly speaking, φ is a collection of smooth functions $\{\varphi_i\}$, each defined on an open set of $U_i \subset X$ corresponding to a trivialization of L . If $s \in H^0(X, L)$, then the norm of s is locally given by $x \mapsto |s_i|^2(x) e^{-\varphi_i(x)}$, where s_i is a local representative of s using the trivialization of L . We can thus perceive $|s| e^{-\varphi}$ as a globally defined function on X . As is well known, we can write the connection ∇ as $\nabla = \nabla' + \nabla''$, where $\nabla'' = \bar{\partial}$, and we can consider the duals of ∇' and ∇'' with respect to the metric φ , and denote them by $(\nabla')^*$ and $(\nabla'')^*$. Then the classical Bochner-Kodaira-Nakano identity states that

$$\nabla'(\nabla')^* + (\nabla')^* \nabla' = \nabla''(\nabla'')^* + (\nabla'')^* \nabla'' + [dd^c \varphi, \Lambda],$$

where one can show that $dd^c \varphi$ is the curvature operator associated with ∇ . In the same way as in this article, this identity can be used to show the solvability of the $\bar{\partial}$ -equation (the argument is basically due to Hörmander [6]):

Theorem 5.16. (Hörmander, [6]) *Let β be a (p, q) -form with values on L such that $\bar{\partial}\beta = 0$ and $\int_X |\beta|^2 e^{-\varphi} dV_X < +\infty$, and assume we can find a metric φ on L such that $([dd^c \varphi, \Lambda]\alpha, \alpha) \geq c\|\alpha\|^2$ for each compactly supported $(p, q+1)$ -form α with values in L . Then we can solve the equation*

$$\bar{\partial}\alpha = \beta$$

with

$$\int_X |\alpha|^2 e^{-\varphi} dV_X \leq \frac{1}{c} \int_X |\beta|^2 e^{-\varphi} dV_X,$$

where dV_X is the volume element on X .

Now, let us instead consider the Laplace operator \square_φ : if $k > n$ and α is a k -form in L_φ^2 we know that (5.7) holds, that is

$$\|d\alpha\|^2 + \|d^*\alpha\|^2 \geq (k-n)\|\alpha\|^2,$$

under the assumption that the metric in question is $dd^\# \varphi$. As we already have seen, the left-hand-side is equal to $\langle \square_\varphi \alpha, \alpha \rangle$, and by the Cauchy-inequality

$$\langle \square_\varphi \alpha, \alpha \rangle \leq \|\square_\varphi \alpha\| \cdot \|\alpha\|.$$

Thus

$$\|\square_\varphi \alpha\| \geq (k-n)\|\alpha\|.$$

Since \square_φ is self-adjoint we can apply Theorem 5.10 to obtain:

Theorem 5.17. *With the notation and assumptions above, we can for each d -closed k -form β solve the equation*

$$\square_\varphi \alpha = \beta,$$

with

$$\|\alpha\|_\varphi^2 \leq (k-n)^2 \|\beta\|_\varphi^2.$$

6. THE LEGENDRE TRANSFORM

We recall the definition and some properties of the Legendre transform:

Definition 6.1. Let f be a convex function on \mathbb{R}^n . The Legendre transform of f is given by

$$(6.1) \quad f^*(y) = \sup_{x \in \mathbb{R}^n} (x \cdot y - f(x)),$$

for $y \in \mathbb{R}^n$.

The Legendre transform of a convex function is again convex, and $(f^*)^* = f$. Let us assume that f is smooth. Then the supremum in (6.1) (if it is not equal to $+\infty$, which we always shall assume in this section) is achieved at a point $x(y)$ for which

$$y = \nabla f(x(y)),$$

where ∇f denotes the gradient of f ; to see this is simply a matter of differentiating the expression inside of the supremum. Thus

$$f^*(y) = x(y) \cdot y - f(x(y)).$$

By a small calculation this implies that if $y = \nabla f(x(y))$ as above, then

$$\nabla f^*(y) = x.$$

Thus, if we consider the map $\psi : \mathbb{R}^n \rightarrow \mathbb{R}^n$, given by

$$\psi(x) = \nabla f(x),$$

then

$$\psi^{-1}(y) = \nabla f^*(y).$$

For any smooth map $\psi : \mathbb{R}^n \rightarrow \mathbb{R}^n$, the pullback, ψ^* , of a form $(p, 0)$ -form is just the regular pullback of a differential p -form, and we extend ψ^* to act on (p, q) -forms by requiring it to be J -linear, that is

$$\psi^* J(\alpha) = J(\psi^* \alpha),$$

for any (p, q) -form α . This makes sense since, in coordinates, this makes for

$$\psi^*(\alpha_{IJ}(x) dx_I \wedge d\xi_J) = \alpha_{IJ}(\psi(x)) \psi^* dx_I \wedge J(\psi^* dx_J),$$

and every (p, q) -form can be written as a linear combination of such forms.

Proposition 6.2. *Let $\psi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a diffeomorphism. Then any integrable (n, n) -form α satisfies,*

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} \alpha = \int_{\mathbb{R}^n \times \mathbb{R}^n} \psi^*(\alpha) / \det(D\psi).$$

Proof. This is a simple consequence of the usual change of variable formula for n -forms on \mathbb{R}^n : let $\alpha = \alpha_0(x) c_n dx \wedge d\xi$. Then $\psi^* \alpha(x) = (\alpha_0 \circ \psi)(x) \cdot (\det(D\psi)(x))^2 c_n dx \wedge d\xi$. Thus

$$\begin{aligned} \int_{\mathbb{R}^n \times \mathbb{R}^n} \psi^*(\alpha) / \det(D\psi) &= \int_{\mathbb{R}^n \times \mathbb{R}^n} (\alpha_0 \circ \psi)(x) \cdot \det(D\psi)(x) c_n dx \wedge d\xi = \\ &= \int_{\mathbb{R}^n} (\alpha_0 \circ \psi)(x) \cdot \det(D\psi)(x) dx = \int_{\mathbb{R}^n} \alpha_0(x) dx = \int_{\mathbb{R}^n \times \mathbb{R}^n} \alpha. \end{aligned}$$

□

Now, let φ be a smooth, strictly convex function, and associate to φ the \mathbb{R} -Kähler form

$$\omega^\varphi = dd^\# \varphi = \sum_{i,j=1}^n \varphi_{ij} dx_i \wedge d\xi_j,$$

with $\varphi_{ij} = \frac{\partial^2 \varphi}{\partial x_i \partial x_j}$. Let also $\psi = \nabla \varphi^*$ which is a diffeomorphism. Since $\nabla \varphi^*$ and $\nabla \varphi$ are inverse to each other by the above, we have $(\varphi_{ij}^*) = (\varphi^{ij})$ where (φ^{ij}) denotes the inverse of the matrix (φ_{ij}) . Thus, since $\psi^* dx_i = \sum_{k=1}^n \varphi_{ik}^* dx_k$ and $\psi^* d\xi_i = \sum_{k=1}^n \varphi_{ik}^* d\xi_k$, we conclude that

$$\psi^* \omega^\varphi = \sum_{i,j,k,l=1}^n \varphi_{ij} \varphi^{ik} \varphi^{jl} dx_k \wedge d\xi_l = \sum_{j,l=1}^n \varphi^{jl} dx_j \wedge d\xi_l.$$

Here we used that the matrix (φ_{ij}) is symmetric so that $\sum_{i=1}^n \varphi_{ij} \varphi^{ik} = \delta_{jk}$. Thus, we obtain:

$$(6.2) \quad \psi^* \omega^\varphi = \omega^{\varphi^*}.$$

Recall from section 2 that the norm of a $(p, 0)$ -form α satisfies the relation

$$|\alpha|_{dd^\# \varphi}^2 \omega_n^\varphi = c_p \alpha \wedge J(\alpha) \wedge \omega_{n-p}^\varphi.$$

Applying ψ^* to both sides of this equality and using (6.2), give us

$$\psi^*(|\alpha|_{dd^\# \varphi}^2 \omega_n^\varphi) = c_p \psi^* \alpha \wedge J(\psi^* \alpha) \wedge \omega_{n-p}^{\varphi^*}.$$

This in turn is equivalent to

$$|\alpha|_{dd^\# \varphi}^2(\psi(x)) \omega_n^{\varphi^*} = |\psi^* \alpha|_{dd^\# \varphi^*}^2(x) \omega_n^{\varphi^*},$$

and we have proved:

Proposition 6.3. *Let α be a $(p, 0)$ -form. Then, at any point x ,*

$$|\psi^* \alpha|_{dd^\# \varphi^*}^2(x) = |\alpha|_{dd^\# \varphi}^2(\psi(x)).$$

Let us consider the integral

$$\int |\alpha|^2 e^{-\varphi} dx \wedge d\xi;$$

we shall see how this integral transform under the Legendre transform of φ under the additional assumption that φ is r -homogeneous, that is, when for each $y \in \mathbb{R}^n$,

$$(6.3) \quad \varphi(ty) = t^r \varphi(y),$$

for $t \geq 0$. Differentiating the relation (6.3) with respect to t , and evaluating at $t = 1$ tells us that,

$$y(\nabla \varphi)(y) = r\varphi(y).$$

This result is sometimes referred to as Euler's theorem on homogeneous functions. Moreover, we know that

$$\varphi^*(x) = y \cdot (\nabla \varphi)(y) - \varphi(y)$$

where y is such that $x = \nabla \varphi(y)$. Thus,

$$\varphi^*(x) = (r-1)\varphi(y).$$

Furthermore,

$$\psi^*(dx \wedge d\xi) = \det(\varphi_{ij}^*)^2 dx \wedge d\xi.$$

Thus, by Proposition 6.2 and Proposition 6.3, we obtain

$$\begin{aligned} \int_{\mathbb{R}^n \times \mathbb{R}^n} |\alpha|_{\omega^\varphi}^2 e^{-\varphi} c_n dx \wedge d\xi &= \int_{\mathbb{R}^n \times \mathbb{R}^n} \psi^*(|\alpha|_{\omega^\varphi}^2 e^{-\varphi} c_n dx \wedge d\xi) / \det(D\psi) = \\ &= \int_{\mathbb{R}^n \times \mathbb{R}^n} |\psi^* \alpha|_{\omega^{\varphi^*}}^2 e^{-\frac{\varphi^*}{r-1}} \det(\varphi_{ij}^*) c_n dx \wedge d\xi = \int_{\mathbb{R}^n \times \mathbb{R}^n} |\psi^* \alpha|_{\omega^{\varphi^*}}^2 e^{-\frac{\varphi^*(y)}{r-1}} \omega_n^{\varphi^*}. \end{aligned}$$

Let us record this result as a proposition:

Proposition 6.4. *Let φ be a r -homogeneous, convex and smooth function, and let $\psi = \nabla \varphi^*$. Then any integrable $(p, 0)$ -form α satisfies*

$$(6.4) \quad \int_{\mathbb{R}^n \times \mathbb{R}^n} |\alpha|_{\omega^\varphi}^2 e^{-\varphi} c_n dx \wedge d\xi = \int_{\mathbb{R}^n \times \mathbb{R}^n} |\psi^* \alpha|_{\omega^{\varphi^*}}^2 e^{-\frac{\varphi^*(y)}{r-1}} \omega_n^{\varphi^*}.$$

Under these circumstances we can prove the following:

Proposition 6.5. *Let $\beta \in L_\phi^2(dd^\# \phi)$ be a $(p, 0)$ -form, with ϕ an r -homogeneous, convex and smooth function, with $r > 1$. Then we can solve*

$$d\alpha = \beta$$

with the estimate

$$(6.5) \quad \int_{\mathbb{R}^n \times \mathbb{R}^n} |\alpha|_{\omega^\phi}^2 e^{-\phi} \omega_n^\phi \leq \frac{1}{p(r-1)} \int_{\mathbb{R}^n \times \mathbb{R}^n} |\beta|_{\omega^\phi}^2 e^{-\phi} \omega_n^\phi.$$

Proof. If $s = \frac{r}{r-1}$, then it is well known that ϕ^* is an s -homogeneous function. Thus, if we let

$$\varphi = (r-1)^{1-r} \phi^*,$$

then φ is s -homogeneous, and satisfies $\frac{\varphi^*}{r-1} = \phi$. To simplify notation we let $\tau = (r-1)^{1-r}$. If α is a $(p, 0)$ -form, then

$$\begin{aligned} |\alpha|_{\omega^\varphi}^2 &= \tau^{-p} |\alpha|_{\omega^{\phi^*}}^2, \\ |\psi^* \alpha|_{\omega^{\varphi^*}}^2 &= (r-1)^{-p} |\psi^* \alpha|_{\omega^\phi}^2, \\ \omega_n^{\varphi^*} &= (r-1)^n \omega_n^\phi; \end{aligned}$$

thus, with $\psi = \nabla \varphi^*$ formula (6.4) transforms into

$$(6.6) \quad \int_{\mathbb{R}^n \times \mathbb{R}^n} |\alpha|_{\omega^{\phi^*}}^2 e^{-\tau \phi^*} c_n dx \wedge d\xi = (r-1)^{n-pr} \int_{\mathbb{R}^n \times \mathbb{R}^n} |\psi^* \alpha|_{\omega^\phi}^2 e^{-\phi} \omega_n^\phi.$$

Now let us consider the form $\gamma = (\psi^{-1})^* \beta$. Then, since $\psi^* (\psi^{-1})^* \beta = \beta$, we see that $\psi^* \gamma \in L_\phi^2(dd^\# \phi)$, and by the above the form γ satisfies

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} |\gamma|_{\omega^{\phi^*}}^2 e^{-\tau \phi^*} c_n dx \wedge d\xi < +\infty.$$

Moreover, γ is d -closed, since $d\psi^* = \psi^* d$. Thus, by Theorem 5.14 we can find a $(p-1, 0)$ -form η such that

$$d\eta = \gamma$$

and

$$\begin{aligned} \tau^{-(p-1)} \int_{\mathbb{R}^n \times \mathbb{R}^n} |\eta|_{\omega^{\phi^*}}^2 e^{-\tau \phi^*} c_n dx \wedge d\xi &\leq \\ &\leq \frac{1}{p} \tau^{-p} \int_{\mathbb{R}^n \times \mathbb{R}^n} |\gamma|_{\omega^{\phi^*}}^2 e^{-\tau \phi^*} c_n dx \wedge d\xi, \end{aligned}$$

where we used that

$$|\alpha|_{\omega^{\tau \phi^*}}^2 = \tau^{-p} |\alpha|_{\omega^{\phi^*}}^2,$$

for any $(p, 0)$ -form α . If we apply formula (6.6) to this inequality, with $\alpha = \psi^* \eta$, then since $\psi^* \gamma = \beta$ we see that

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} |\alpha|_{\omega^\phi}^2 e^{-\phi} \omega_n^\phi \leq \frac{1}{p(r-1)} \int_{\mathbb{R}^n \times \mathbb{R}^n} |\beta|_{\omega^\phi}^2 e^{-\phi} \omega_n^\phi.$$

Also,

$$d\alpha = d\psi^* \eta = \psi^* \gamma = \beta,$$

which concludes the proof. \square

While the estimate of the above proposition is similar to that of Theorem 5.14, it does not seem to follow from the previous formalism in a direct way.

From section 5 we know that in order to obtain existence theorems for the d -operator we need to examine the commutator term $[dd^\# \varphi, \Lambda]$, and we used in that section the choice $\omega = dd^\# \varphi$ for φ a smooth, strictly convex function, to obtain $[dd^\# \varphi, \Lambda] \alpha = (k-n) \alpha$, for α a k -form. Let us instead assume that ϕ is a smooth, strictly concave function. Then $-dd^\# \phi$ is a closed positive $(1, 1)$ -form and we can let $\omega^\phi = -dd^\# \phi$ be our Kähler form. In this situation we see that

$$[dd^\# \phi, \Lambda] \alpha = (n-k) \alpha,$$

and thus, we have

$$||d^* \alpha|_\phi^2 + ||d\alpha|_\phi^2 \geq (n-k) ||\alpha|_\phi^2,$$

for every smooth, compactly supported form α , where $||\alpha||_\phi^2 = \int_{\mathbb{R}^n \times \mathbb{R}^n} |\alpha|_{\omega^\phi}^2 e^{-\phi} \omega_n^\phi$, and the dual of d is with respect to this norm. Applying Theorem 5.10 we obtain the following:

Proposition 6.6. *Let ϕ be a smooth, strictly concave function. Then, for every closed $(p, 0)$ -form $\beta \in L_\phi^2$, we can solve the equation*

$$d\alpha = \beta,$$

with

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} |\alpha|_{\omega^\phi}^2 e^{-\phi} \omega_n^\phi \leq \frac{1}{n-p} \int_{\mathbb{R}^n \times \mathbb{R}^n} |\beta|_{\omega^\phi}^2 e^{-\phi} \omega_n^\phi.$$

It is interesting to compare this result with Proposition 6.5: let φ be 2-homogeneous and assume that $\beta \in L_\varphi^2 \cap L_\phi^2$. Then we can find solutions α_1 and α_2 to $d\alpha = \beta$ such that

$$\begin{aligned} \int_{\mathbb{R}^n \times \mathbb{R}^n} |\alpha_1|_{\omega^\varphi}^2 e^{-\varphi} \omega_n^\varphi &\leq \frac{1}{p} \int_{\mathbb{R}^n \times \mathbb{R}^n} |\beta|_{\omega^\varphi}^2 e^{-\varphi} \omega_n^\varphi \\ \int_{\mathbb{R}^n \times \mathbb{R}^n} |\alpha_2|_{\omega^\phi}^2 e^{-\phi} \omega_n^\phi &\leq \frac{1}{n-p} \int_{\mathbb{R}^n \times \mathbb{R}^n} |\beta|_{\omega^\phi}^2 e^{-\phi} \omega_n^\phi. \end{aligned}$$

Thus, we can solve the equation $d\alpha = \beta$ with fundamentally different estimates on the solutions: in one case the weight φ is convex and in the other ϕ is concave.

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